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Abel-like differential equations with no periodic solutions ☆

J.L. Bravo [a,*], J. Torregrosa [b]

[a] *Departamento de Matematicas, Universidad de Extremadura, 06071 Badajoz, Spain*
[b] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*

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Abstract

We present various criteria for the non-existence of positive periodic solutions of generalized Abel differential equations with periodic coefficients that can change sign. As an application, we obtain some families of planar vector fields without limit cycles.

Keywords: Abel equation; Limit cycles; Hilbert's 16th problem; Rigid systems; Periodic solutions

1. Introduction

Hilbert's 16th problem [12] is usually stated as determining the maximum number of limit cycles (isolated periodic orbits) in terms of the degrees of a polynomial system in the plane

$$\begin{cases} x' = P(x, y), \\ y' = Q(x, y), \end{cases} \tag{1.1}$$

where P and Q are polynomials. Although there has long been intense research interest in this problem, only recently has it been proved that the number of limit cycles is finite for each individual equation [6,13].

Bounds on the number of limit cycles have only been found for some families of polynomial systems, the problems most extensively studied being non-existence and uniqueness. In most cases, a change of variables proposed by Cherkas [4] is used to obtain an equivalence between the number of limit cycles of (1.1) for some P and Q, and the number of positive periodic solutions of an Abel-like differential equation

$$x' = \sum_{i=0}^{n} A_i(t) x^i, \tag{1.2}$$

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* Corresponding author.
E-mail addresses: trinidad@unex.es (J.L. Bravo), torre@mat.uab.cat (J. Torregrosa).

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for certain A_i being functions in $\sin(t)$ and $\cos(t)$. Examples of this transform can be found in [3,5,15]. In all the paper we will assume $A_0 = 0$, then $u(t) \equiv 0$ is always a solution of (1.2), corresponding with the origin in the planar system, which is assumed to be a critical point of (1.1).

For $n = 3$ (the Abel differential equation), Lins Neto [14] showed that (1.2) may have any number of periodic solutions, and Shahshahani [16] proved that if $A_3(t)$ does not change sign then (1.2) has at most three periodic solutions. Gasull and Llibre [8] proved that if $A_2(t)$ does not change sign then (1.2) has at most three periodic solutions. Álvarez, Gasull and Giacomini [1] proved that if $A_1(t) \equiv 0$ and there exits $a, b \in \mathbb{R}$ such that $aA_2(t) + bA_3(t)$ has definite sign then (1.2) has at most three periodic solutions.

The existence or non-existence of an isolated periodic positive solution in generalized Abel equations has also been studied in recent work [2,11,17]. All the aforementioned studies, however, were based on the constancy of sign of some of the functions involved in (1.2). The main goal of the present communication is to show new technics to deal with functions with some "change of sign."

First, we need to precise the notion of "change of sign": A non-identically zero function f will be said to change sign at t_0 if there exist $t_1 < t_0 < t_2$ such that $f(t)f(s) \leqslant 0$ for all $t \in (t_1, t_0)$, $s \in (t_0, t_2)$, and the inequality is strict for some t and s.

We shall here study some families of Abel-like equations of any degree, for which all the coefficients change sign. In particular, we consider

$$x' = f(t)x^{\alpha} + h(t)x^{\beta}, \quad t \in [0, 2\pi], \tag{1.3}$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$, $\alpha, \beta \geqslant 1$, f, h 2π-periodic continuous functions, $h(t)$ with at most four changes of sign in $[0, 2\pi)$. We shall look for conditions on f and h such that (1.3) has no isolated positive periodic solutions. As we are only interested in positive solutions, we consider (1.3) when $x \geqslant 0$. Therefore, it is well defined for any $\alpha, \beta \geqslant 1$, and there are uniqueness of solutions. Note that if $\alpha, \beta \leqslant 1$, after the change of variables $x \to x^{-1}$, it holds that $\alpha, \beta \geqslant 1$.

We shall distinguish three cases: h with zero, two or four changes of sign, summarized in Theorems 2.1, 2.4 and 2.8, respectively. The boundedness of the number of limit cycles of generalized Abel equations has also been studied in [2,7], but when the exponents are natural and one of the coefficients does not change sign.

Eq. (1.3) was motivated by rigid planar polynomial systems, i.e., equations of the form

$$\begin{cases} x' = y + xF(x, y), \\ y' = -x + yF(x, y), \end{cases} \tag{1.4}$$

where F is a polynomial. The study of the limit cycles of these systems has also been considered in [9,10] for a particular choice of F.

Transforming to polar coordinates, one obtains that the number of limit cycles of (1.4) is the same as the number of positive periodic solutions of

$$r' = rF(r\cos\theta, r\sin\theta). \tag{1.5}$$

Note that a term (monomial) of F in (1.4) contributes to (1.5) as a term of the form $Kr^{m+1}\cos^k\theta\sin^{m-k}\theta$ for some constant K. These terms have zero, two, or four zeros for $\theta \in [0, 2\pi)$. In Section 3, we provide some families of rigid planar vector fields such that the associated Abel-like equation satisfies the hypotheses of one of Theorems 2.1, 2.4, or 2.8. In particular, Theorem 3.4 sets that if

$$F(x, y) = ax^{k_1}y^{n-k_1} + bx^{k_2}y^{n-k_2} + cx^{k_3}y^{m-k_3},$$

n is odd, and m is even, then (1.4) either has no limit cycles or is a center.

Eq. (1.4) when F is a quadratic polynomial and the coefficient of the main term does not change sign is described fully in [9,10], while the case with coefficients of the main term changing sign remains open. Some families belonging to this case are studied in Section 3.

2. Main results

First, suppose that h does not change sign. Next result set sufficient conditions for (1.3) having no positive periodic solution.

Theorem 2.1. *Let $\alpha, \beta \geqslant 1$, $\alpha \neq \beta$, and assume that f, h are 2π-periodic continuous functions such that*

$$\int_0^{2\pi} f(t)\,dt = 0,$$

and h has no changes of sign. Then (1.3) *has no positive periodic solutions if and only if $h(t) \not\equiv 0$.*

Proof. Consider $h(t) \not\equiv 0$, and the perturbed differential equation

$$x' = f(t)x^\alpha + \lambda h(t)x^\beta. \tag{2.1}$$

For $\lambda = 0$, one has

$$x' = f(t)x^\alpha,$$

an equation in separate variables. Let u be a bounded solution. Integrating over $[0, 2\pi]$, one obtains

$$\int_0^{2\pi} \frac{u'(t)}{u^\alpha(t)}\,dt = \int_0^{2\pi} f(t)\,dt = 0.$$

By a change of variables,

$$\int_{u(0)}^{u(2\pi)} \frac{1}{x^\alpha(t)}\,dx = 0.$$

Since any primitive of the integrated function is monotonic, then $u(2\pi) = u(0)$.

Let $u(t, x, \lambda)$ ($u(t)$ when no confusion is possible) denote the solution of (2.1) determined by $u(0, x, \lambda) = x$.

Suppose that $u(t, x, \lambda)$ is well defined at $t = 2\pi$ for every $0 \leqslant \lambda \leqslant 1$. Differentiating (2.1) with respect to λ, one obtains

$$u_\lambda'(t) = \alpha f(t)u^{\alpha-1}(t)u_\lambda(t) + \beta\lambda h(t)u^{\beta-1}(t)u_\lambda(t) + h(t)u^\beta(t).$$

Integrating this equation over $[0, 2\pi]$ gives

$$u_\lambda(2\pi) = \int_0^{2\pi} \left(h(t)u^\beta(t)e^{\int_t^{2\pi}(\alpha f(s)u^{\alpha-1}(s)+\beta\lambda h(s)u^{\beta-1}(s))\,ds}\right)dt > 0.$$

Since $u(2\pi) = u(0)$ for $\lambda = 0$, then $u(2\pi) > u(0)$ for every $0 \leqslant \lambda \leqslant 1$.

Let prove that if $u(t, x, 1)$ is well defined at $t = 2\pi$, then $u(t, x, \lambda)$ is well defined at $t = 2\pi$ for every $0 \leqslant \lambda \leqslant 1$. Therefore, (1.3) has no positive periodic solution for $\lambda = 1$.

By the change of variables $t \to -t$, one may assume that $h(t) \geqslant 0$. Therefore, if $u(t, x, 1)$ is well defined at $t = 2\pi$, then $u(t, x, 1) \geqslant u(t, x, \lambda) \geqslant 0$ for every $\lambda \leqslant 1$, and then $u(t, x, \lambda)$ is well defined at $t = 2\pi$.

Finally, if $h(t) \equiv 0$, then (1.3) is an equation in separate variables and one can check that every positive bounded solution is periodic. $\square$

From the previous proof one deduces:

Corollary 2.2. *If h does not change sign, is not identically zero, and u is a bounded solution of* (1.3), *then*

$$\operatorname{sign}\big(u(2\pi) - u(0)\big) = \operatorname{sign}(h).$$

The following result is a generalization of Theorem 2.1 when there are more summands in (1.3).

Corollary 2.3. *Let $1 \leqslant \alpha < \beta_1, \dots, \beta_n \in \mathbb{R}$, and assume that f, h_i, $i = 1, \dots, n$, are 2π-periodic continuous functions such that*

$$\int_0^{2\pi} f(t)\,dt = 0,$$

and for every $i = 1, \dots, n$, h_i has no changes of sign, $h_i(t) \not\equiv 0$, and $h_i(t) \geqslant 0$. Then

$$x' = f(t)x^\alpha + \sum_{i=1}^n h_i(t)x^{\beta_i}$$

has no positive periodic solutions.

The result also holds if $\alpha > \beta_1, \dots, \beta_n \in \mathbb{R}$ for every $i \in \{1, \dots, n\}$, or $h_i(t) \leqslant 0$ for every $i \in \{1, \dots, n\}$.

Proof. Proceeding as in the proof of Theorem 2.1, one obtains

$$u_\lambda(2\pi) = \int_0^{2\pi} \left(\sum_{i=1}^n h_i(t)u^{\beta_i}(t)e^{\int_t^{2\pi}(\alpha f(s)u^{\alpha-1}(s)+\sum_{i=1}^n \beta_i \lambda h_i(s)u^{\beta_i-1}(s))\,ds} \right) dt,$$

which again is positive, and the proof follows identically. □

The next two theorems can be applied when the function h changes sign. First, rewrite (1.3) as

$$x' = \big(f(t) + g(t)\big)x^\alpha + h(t)x^\beta, \quad t \in [0, 2\pi]. \tag{2.2}$$

We shall prove that, under appropriate assumptions, there are no positive periodic solutions.

Theorem 2.4. *Let $\alpha, \beta \geqslant 1$, $\alpha \neq \beta$, and assume that f, g, h are 2π-periodic continuous functions satisfying*

(i) *$f(t - \pi/2)$ is odd and f does not change sign in $(\pi/2, 3\pi/2)$.*
(ii) *g is odd.*
(iii) *h is odd and does not change sign in $(0, \pi)$.*

Then (2.2) *has no positive periodic solutions.*

Remark. With these hypothesis the functions f and h change sign exactly two times in each period. Moreover, since f and h are 2π-periodic, then $f(t - 3\pi/2)$ and $h(t - \pi)$ are odd, so after the changes of variables $t \to -t$, $t \to \pi - t$ and $t \to \pi + t$, (2.2) satisfies (i)–(iii).

In order to prove Theorem 2.4, we perturb (2.2) multiplying f by a parameter λ,

$$x' = \big(\lambda f(t) + g(t)\big)x^\alpha + h(t)x^\beta. \tag{2.3}$$

We shall prove that (2.3) is a center for $\lambda = 0$ and that all positive periodic solutions disappear as λ is increased. In particular, for $\lambda = 1$, (2.2) has no positive periodic solutions.

Proposition 2.5. *Assume that $\lambda = 0$. Then every positive bounded solution of* (2.3) *is 2π-periodic.*

Proof. Let u be a positive bounded solution of (2.3). Then u is even because g and h are odd. Therefore, $u(\pi) = u(-\pi)$ and u is periodic. □

Now, we are on conditions for proving the theorem.

Proof of Theorem 2.4. Assume that $f(t), h(t) \geqslant 0$ for all $t \in [0, \pi/2]$, and $\alpha < \beta$. Note that this determines the sign of f and h for all $t \in [0, 2\pi]$ and, by periodicity, for all $t \in \mathbb{R}$. The rest of the cases are analogous.

Let $u(t,x,\lambda)$ denote the solution of (2.3) determined by $u(0,x,\lambda)=0$, and $u_\lambda(t,x,\lambda)$ denote the derivative of $u(t,x,\lambda)$ with respect to λ. To simplify the notation, we shall write $u(t)$ and $u_\lambda(t)$ when this does not lead to confusion.

Suppose that $u(t,x,\lambda)$ is well defined at $t=\pm\pi$ for every $0\leqslant\lambda\leqslant 1$. We shall prove that $u_\lambda(\pi,x,\lambda)>0$ and $u_\lambda(-\pi,x,\lambda)<0$. Then

$$u(\pi,x,\lambda)-u(-\pi,x,\lambda)>0 \quad \text{for every } \lambda>0.$$

In particular, for $\lambda=1$ it implies that (2.2) has no positive periodic solutions.

Differentiating (2.3) with respect to λ, one obtains

$$u_\lambda'(t)=f(t)u^\alpha(t)+\alpha\big(\lambda f(t)+g(t)\big)u^{\alpha-1}(t)u_\lambda(t)+\beta h(t)u^{\beta-1}(t)u_\lambda(t),$$

and integrating over $[0,\pi]$,

$$\begin{aligned}
u_\lambda(\pi)&=\int_0^\pi f(t)u^\alpha(t)e^{\int_t^\pi(\alpha(\lambda f(s)+g(s))u^{\alpha-1}(s)+\beta h(s)u^{\beta-1}(s))\,ds}\,dt\\
&=\int_0^\pi f(t)u^\alpha(t)e^{\int_t^\pi(\alpha u'(s)/u(s)+(\beta-\alpha)h(s)u^{\beta-1}(s))\,ds}\,dt\\
&=\int_0^\pi f(t)u^\alpha(t)u^\alpha(\pi)/u^\alpha(t)e^{\int_t^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\,dt\\
&=u^\alpha(\pi)\int_0^\pi f(t)e^{\int_t^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\,dt.
\end{aligned}$$

Note that $u^\alpha(\pi)$ is positive, so that the sign of $u_\lambda(\pi)$ is the sign of the integral. Since $f(t-\pi/2)$ is odd, one may rewrite the foregoing integral as

$$\int_0^\pi f(t)e^{\int_t^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\,dt=\int_0^{\pi/2} f(t)\big(e^{\int_t^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}-e^{\int_{\pi-t}^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\big)\,dt.$$

Now we shall prove that the expression in parentheses is positive for all $t\in[0,\pi/2]$, so that $u_\lambda(\pi)>0$. One may rewrite that expression as

$$\begin{aligned}
&e^{\int_t^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}-e^{\int_{\pi-t}^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\\
&\quad=e^{\int_{\pi/2}^\pi(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\big(e^{\int_t^{\pi/2}(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}-e^{-\int_{\pi/2}^{\pi-t}(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\big).
\end{aligned}$$

Since $\beta-\alpha>0$, the sign of this expression is the same as the sign of

$$\int_t^{\pi/2}h(s)u^{\beta-1}(s)\,ds+\int_{\pi/2}^{\pi-t}h(s)u^{\beta-1}(s)\,ds>0 \tag{2.4}$$

as $u^{\beta-1}$ and h are positive in $[0,\pi]$.

Finally, we shall prove that $u_\lambda(-\pi)<0$. By the change of variables $t\to -t$ in (2.3), and since g and h are odd,

$$u'(-t)=\big(-\lambda f(-t)+g(t)\big)u^\alpha(-t)+h(t)u^\beta(-t).$$

Then $u(-t,\lambda)=\bar u(t,-\lambda)$, where $\bar u$ denotes the solutions of (2.3) for $f(-t)$ instead of $f(t)$. Note that (i) implies that $f(-t+\pi/2)$ is odd

$$f(-t+\pi/2)=f\big((-t+2\pi/2)-\pi/2\big)=-f(t-2\pi/2-\pi/2)=-f(t+\pi/2).$$

Now, since $f(-t+\pi/2)$ is odd and the sign of λ does not affect the foregoing arguments, $u_\lambda(-t,\lambda)=-\bar u_\lambda(t,-\lambda)<0$.

Now, suppose that $u(t,x,1)$ is a bounded solution of (2.3). We shall prove that we may assume that $u(t,x\lambda)$ is defined for $t=\pm\pi$, $0\leqslant\lambda\leqslant 1$. Therefore, by arguments above, (2.3) has no positive periodic solutions.

By the changes $t\to -t$, $t\to\pi+t$ or $t\to\pi-t$, one may assume that $f(t)\geqslant 0$ for every $t\in[0,\pi/2]$, and $h(t)(\beta-\alpha)\geqslant 0$ for every $t\in[0,\pi/2]$. Then, if $u(t,x,\lambda)$ is defined, and $0\leqslant t\leqslant\pi/2$, as $f(t)\geqslant 0$, then $u_\lambda(t,x,\lambda)\geqslant 0$. If $\pi/2\leqslant t\leqslant\pi$, then

$$u_\lambda(t,x,\lambda)=\int_0^{\pi-t} f(s)u^\alpha(s)e^{\int_s^\pi(\alpha(\lambda f(\tau)+g(\tau))u^{\alpha-1}(\tau)+\beta h(\tau)u^{\beta-1}(\tau))\,d\tau}\,ds$$

$$+\int_{\pi-t}^{t} f(s)u^\alpha(s)e^{\int_s^\pi(\alpha(\lambda f(\tau)+g(\tau))u^{\alpha-1}(\tau)+\beta h(\tau)u^{\beta-1}(\tau))\,d\tau}\,ds.$$

The first integral is positive since $f(s)\geqslant 0$ for $s\in(0,\pi-t)$, and arguing as for $u(\pi,x,\lambda)$, one obtains that the second integral is positive, so $u_\lambda(t,x,\lambda)\geqslant 0$.

Then, $0<u(t,x,\lambda)<u(t,x,1)$ for every $0\leqslant\lambda\leqslant 1$, $0<t\leqslant\pi$, as long as both solutions are defined. As a consequence, $u(\pi,x,\lambda)$ is defined for every $0\leqslant\lambda\leqslant 1$. In particular $u(\pi,x,0)$ is defined. Since $u(t,x,0)$ is even, $u(-\pi,x,0)$ is defined. Now, using that $u_\lambda(-t,x,\lambda)=-u_\lambda(t,x,-\lambda)<0$, $0<u(t,x,\lambda)<u(t,x,0)$ for every $0\leqslant\lambda\leqslant 1$, $-\pi\leqslant t<0$, as long as both solutions are defined. Then $u(-\pi,x,\lambda)$ is defined for every $0\leqslant\lambda\leqslant 1$. $\square$

The next result follows from the previous proof.

Corollary 2.6. *Assume that the hypothesis of Theorem* 2.4 *holds and u is a positive bounded solution of* (2.2)*, then*

$$\operatorname{sign}\big(u(2\pi)-u(0)\big)=\operatorname{sign}\big((\beta-\alpha)f(t)h(s)\big)$$

for any $t,s\in(0,\pi/2)$ such that $f(t),h(s)\neq 0$.

As a generalization of Theorem 2.4, a similar result can be obtained when (2.2) has more terms.

Corollary 2.7. *Let $1\leqslant\alpha<\beta_1,\dots,\beta_n\in\mathbb{R}$, and assume that f, g, h_i, $i=1,\dots,n$, are 2π-periodic continuous functions such that $f(t-\pi/2)$ is odd, f does not change sign in $(\pi/2,3\pi/2)$, g is odd and for every $i=1,\dots,n$, h_i is odd, and $h_i(t)\geqslant 0$ for all $t\in(0,\pi)$. Then*

$$x'=\big(f(t)+g(t)\big)x^\alpha+\sum_{i=1}^n h_i(t)x^{\beta_i}$$

has no positive periodic solutions.

The result also holds if $\alpha>\beta_1,\dots,\beta_n\in\mathbb{R}$ for every $i\in\{1,\dots,n\}$, or $h_i(t)\leqslant 0$ for all $t\in(0,\pi)$ and every $i\in\{1,\dots,n\}$.

Proof. Proceeding as in the proof of Theorem 2.4, one obtains that (2.4) becomes

$$\sum_{i=1}^n\left(\int_t^{\pi/2} h_i(s)u^{\beta_i-1}(s)\,ds+\int_{\pi/2}^{\pi-t} h_i(s)u^{\beta_i-1}(s)\,ds\right)>0,$$

and the proof follows identically. $\square$

Now, suppose that h has four zeros in $[0,2\pi)$. Next result gives sufficient conditions for (2.2) having no positive periodic solution.

Theorem 2.8. *Let $\alpha,\beta\geqslant 1$, $\alpha\neq\beta$, and assume that f, g, h are 2π-periodic continuous functions satisfying*:

(i) *$f(t-\pi/2)$ is odd and f does not change sign in $(\pi/2,3\pi/2)$.*

(ii) *g is odd and $g(t)+g(\pi-t)$ does not change sign in $(0,\pi/2)$, and $g(t)+g(\pi-t)\not\equiv 0$.*
(iii) *h and $h(t-\pi/2)$ are odd and h does not change sign in $(0,\pi/2)$.*

Then (2.2) *has no positive periodic solutions.*

Remark. The condition $g(t)+g(\pi-t)$ does not change sign in $(0,\pi/2)$ is implied by g having definite sign in $[0,\pi]$. Note that after the changes of variables $t\to -t$, $t\to\pi-t$ and $t\pi+t$, (2.2) satisfies (i)–(iii).

Again we shall consider (2.3) and prove that (2.3) is a center for $\lambda=0$ and that all positive periodic solutions disappear when $\lambda\neq 0$.

Proposition 2.9. *Assume that $\lambda=0$. Then every positive bounded solution of* (2.3) *is 2π-periodic.*

Proof. Let u be a bounded solution of (2.3). As a consequence of g and h are odd, u is even. Therefore $u(\pi)=u(-\pi)$, and u is periodic. □

We shall also need to prove that the inequality satisfied by $g(t)+g(\pi-t)$ induces a similar inequality on the solutions.

Proposition 2.10. *Let u be a positive solution of* (2.3)*. Then $u(\pi-t)-u(t)$ does not change sign for all $t\in[0,\pi/2]$. Moreover, the sign is the same as $g(t)+g(\pi-t)$.*

The same result holds for $u(t-\pi)-u(-t)$, $t\in[0,\pi/2]$.

Proof. Assume that $g(t)+g(\pi-t)\geqslant 0$ (the case $g(t)+g(\pi-t)\leqslant 0$ is analogous). By (2.3), and since $f(t-\pi/2)$ and $h(t-\pi/2)$ are odd, one has

$$\begin{aligned} u'(\pi-t) &= -\big(\lambda f(\pi-t)+g(\pi-t)\big)u^{\alpha}(\pi-t)-h(\pi-t)u^{\beta}(\pi-t)\\ &= \big(\lambda f(t)-g(\pi-t)\big)u^{\alpha}(\pi-t)+h(t)u^{\beta}(\pi-t). \end{aligned}$$

Let $v(t)$ be the solution of

$$x'=\big(\lambda f(t)-g(\pi-t)\big)x^{\alpha}+h(t)x^{\beta}$$

determined by $v(\pi/2)=u(\pi/2)$. Since $g(t)+g(\pi-t)\geqslant 0$, $g(t)\geqslant -g(\pi-t)$. Therefore v is a subsolution of (2.3) (i.e., $v'<(\lambda f(t)+g(t))v^{\alpha}+h(t)v^{\beta}$) such that $v(\pi/2)=u(\pi/2)$. Then $u(t)<v(t)=u(\pi-t)$ for all $t<\pi/2$. □

Proof of Theorem 2.8. Assume that $\alpha<\beta$ and

$$f(t),h(t),g(t)+g(\pi-t)\geqslant 0\quad\text{for every } t\in[0,\pi/2].$$

Note that this determines the sign of $f(t)$, $g(t)+g(\pi-t)$, and $h(t)$ for all $t\in[0,2\pi]$ and, by periodicity, for all $t\in\mathbb{R}$. The rest of the cases are analogous.

We shall prove that $u_\lambda(\pi,x,\lambda)<0$ and $u_\lambda(-\pi,x,\lambda)>0$. Then $u(\pi,x,\lambda_1)-u(-\pi,x,\lambda_1)<0$ for every $\lambda_1>0$ such that $u(t,x,\lambda)$ is well defined at $t=\pm\pi$ for every $0\leqslant\lambda$. Arguing as in the proof of Theorem 2.8, it implies that (2.2) has no positive periodic solutions for $\lambda=1$.

Differentiating (2.3) with respect to λ, and integrating over $[0,\pi]$, one has

$$u_\lambda(\pi)=u^{\alpha}(\pi)\int_0^{\pi} f(t)e^{\int_t^{\pi}(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\,dt.$$

As $f(t-\pi/2)$ is odd, one may rewrite the foregoing integral as

$$\int_0^{\pi} f(t)e^{\int_t^{\pi}(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\,dt=\int_0^{\pi/2} f(t)\big(e^{\int_t^{\pi}(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}-e^{\int_{\pi-t}^{\pi}(\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\big)\,dt.$$

Now we shall prove that the expression in parentheses is negative for all $t \in [0, \pi/2]$, so that $u_\lambda(\pi) < 0$. Rewrite that expression as

$$e^{\int_t^\pi (\beta-\alpha)h(s)u^{\beta-1}(s)\,ds} - e^{\int_{\pi-t}^\pi (\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}$$
$$= e^{\int_{\pi/2}^\pi (\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\bigl(e^{\int_t^{\pi/2} (\beta-\alpha)h(s)u^{\beta-1}(s)\,ds} - e^{-\int_{\pi/2}^{\pi-t} (\beta-\alpha)h(s)u^{\beta-1}(s)\,ds}\bigr).$$

Since $\beta - \alpha > 0$, the sign of this expression is the same as the sign of

$$\int_t^{\pi/2} h(s)u^{\beta-1}(s)\,ds + \int_{\pi/2}^{\pi-t} h(s)u^{\beta-1}(s)\,ds.$$

As $h(t - \pi/2)$ is odd, the foregoing expression may be rewritten as

$$\int_t^{\pi/2} h(s)\bigl(u^{\beta-1}(s) - u^{\beta-1}(\pi - s)\bigr)\,ds. \tag{2.5}$$

By Proposition 2.10, $u(s) < u(\pi - s)$. Therefore the above expression is negative.

Finally, as in the proof of Theorem 2.4, using the change of variable $t \to -t$, one obtains

$$u_\lambda(-t, \lambda) = -\bar{u}_\lambda(t, -\lambda) > 0,$$

where $\bar{u}$ denotes the solutions of (2.3) for $f(-t)$ instead of $f(t)$. □

The next result follows from the previous proof.

Corollary 2.11. *If the hypothesis of Theorem* 2.8 *holds and u is a positive bounded solution of* (2.2)*, then*

$$\operatorname{sign}\bigl(u(2\pi) - u(0)\bigr) = \operatorname{sign}\bigl(-f(t_1)\bigl(g(\pi - t_2) + g(t_2)\bigr)h(t_3)(\beta - \alpha)\bigr)$$

for any $t_1, t_2, t_3 \in (0, \pi/2)$ such that $f(t_1)$, $g(\pi - t_2) + g(t_2), h(t_3) \neq 0$.

As in the previous cases, Theorem 2.8 can be generalized to equations with more terms of higher degree.

Corollary 2.12. *Let $1 \leqslant \alpha < \beta_1, \dots, \beta_n \in \mathbb{R}$, and assume that f, g, h_i, $i = 1, \dots, n$, are 2π-periodic continuous functions such that $f(t - \pi/2)$ is odd, f does not change sign in $(\pi/2, 3\pi/2)$, g is odd, $g(t) + g(\pi - t)$ does not change sign in $(0, \pi/2)$, $g(t) + g(\pi - t) \not\equiv 0$, and for every $i = 1, \dots, n$, h_i, $h_i(t - \pi/2)$ are odd, and $h_i(t) \geqslant 0$ for all $t \in (0, \pi/2)$. Then*

$$x' = \bigl(f(t) + g(t)\bigr)x^\alpha + \sum_{i=1}^n h_i(t)x^{\beta_i}$$

has no positive periodic solutions.

The result also holds if $\alpha > \beta_1, \dots, \beta_n \in \mathbb{R}$ for every $i \in \{1, \dots, n\}$, or $h_i(t) \leqslant 0$ for all $t \in (0, \pi/2)$ and every $i \in \{1, \dots, n\}$.

Proof. Proceeding as in the proof of Theorem 2.8, (2.5) becomes

$$\sum_{i=1}^n \int_t^{\pi/2} h_i(s)\bigl(u^{\beta_i-1}(s) - u^{\beta_i-1}(\pi - s)\bigr)\,ds.$$

Since the sign of each of the summands is positive, the whole expression is positive and the proof follows identically. □

3. Applications to rigid planar polynomial vector fields

In the following, we present some families of rigid planar polynomial vector fields

$$\begin{cases} x' = y + xF(x, y), \\ y' = -x + yF(x, y), \end{cases} \tag{3.1}$$

such that, after transforming to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

$$r' = rF(r\cos\theta, r\sin\theta) \tag{3.2}$$

satisfies Theorems 2.1, 2.4, or 2.8, i.e., conditions are imposed on F such that (3.1) has no limit cycles. Equations of this type are considered in [2], with some of the coefficients having constant sign, and studying the existence of exactly one limit cycle and the non-existence of limit cycles. Our results extend the non-existence case.

Write $F(x, y)$ as

$$F(x, y) = \sum_{i,j} c_{ij} x^i y^j.$$

Then (3.2) becomes

$$r' = \sum_{i,j} c_{ij} \cos^i\theta \sin^j\theta r^{i+j+1}. \tag{3.3}$$

Consider one summand, $\phi(\theta) = c_{ij}\cos^i\theta\sin^j\theta r^{i+j+1}$. There are four possibilities for the changes of sign of ϕ in $[0, 2\pi)$:

(1) If i and j are odd, then ϕ does not change sign.
(2) If i is odd and j is even, then $\phi(t - \pi/2)$ is odd and does not change sign in $(\pi/2, 3\pi/2)$.
(3) If i is even and j is odd, then ϕ is odd and does not change sign in $(0, \pi)$.
(4) Finally, if i and j are even, then ϕ, $\phi(t - \pi/2)$ are odd and ϕ does not change sign in $(0, \pi/2)$.

Proposition 3.1. *The family of rigid planar vector fields*

$$\begin{cases} x' = y + x\left(F_n(x, y) + \sum_k c_k x^{2i_k} y^{2j_k}\right), \\ y' = -x + y\left(F_n(x, y) + \sum_k c_k x^{2i_k} y^{2j_k}\right), \end{cases} \tag{3.4}$$

when F_n is a homogeneous polynomial of degree n, n odd, $2i_k + 2j_k > n$, and $c_k \geqslant 0$ for every k, the inequality being strict for some k, has no limit cycles.

The same result holds if $2i_k + 2j_k < n$ for every k or $c_k \leqslant 0$ for every k and the inequality is strict for some k.

Proof. We shall prove that (3.3) satisfies Corollary 2.3.

First, since n is odd, then all the terms of $F_n(r\cos\theta, r\sin\theta)$ are of odd degree in $\cos\theta$ and even in $\sin\theta$, or odd in $\sin\theta$ and even in $\cos\theta$. Thus, we may write

$$rF_n(r\cos\theta, r\sin\theta) = \big(f(\theta) + g(\theta)\big)r^{n+1},$$

where f is the sum of terms odd in $\cos\theta$ and even in $\sin\theta$ and g is the sum of terms odd in $\sin\theta$ and even in $\cos\theta$. Since f, g are periodic, and $f(t - \pi/2)$ and g are odd then

$$\int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} g(t)\,dt = 0.$$

Finally, let $h_k(\theta) = c_k\cos^{2i_k}\theta\sin^{2j_k}\theta$. Then h_k has no changes of sign and, since $c_k \geqslant 0$, $h_k(\theta) \geqslant 0$. Therefore (3.3) satisfies Corollary 2.3 and has no positive periodic solutions and hence (3.4) has no limit cycles. □

Proposition 3.2. *Let n be odd, and consider*

$$F(x,y)=\sum_{k=0}^{(n+1)/2}\left(a_k x^{2k+1}y^{n-2k-1}+b_k x^{n-2k-1}y^{2k+1}\right)+\sum_{2i+2j+1>n}c_{ij}x^{2i+1}y^{2j}.$$

Assume that $a_k, c_{ij}\geqslant 0$ for every k and every i, j, with the inequalities being strict for some i, j, k. Then the family of rigid planar vector fields (1.4) *has no limit cycles.*

The same result holds if the second sum is over $2i+2j+1<n$, $a_k\leqslant 0$ for every k or $c_{ij}\leqslant 0$ for every i, j, the last two inequalities being strict for some indices.

Proof. We shall prove that, under these assumptions, (3.3) satisfies Corollary 2.7.

Changing (1.4) to polar coordinates, one obtains

$$r'=\left(f(\theta)+g(\theta)\right)r^{n+1}+\sum_{2i+2j+1>n}k_{ij}(\theta)r^{2i+2j+2},$$

where

$$f(\theta)=\sum_{k=0}^{(n+1)/2}a_k\cos^{2k+1}\theta\sin^{n-2k-1}\theta,$$

$$g(\theta)=\sum_{k=0}^{(n+1)/2}b_k\cos^{n-2k-1}\theta\sin^{2k+1}\theta=\sin\theta\sum_{k=0}^{(n+1)/2}b_k\cos^{n-2k-1}\theta\sin^{2k}\theta,$$

and

$$h_{ij}(\theta)=c_{ij}\cos^{2i+1}\theta\sin^{2j}\theta\quad\text{for every } 2i+2j+1>n.$$

The function f is the sum of monomials odd in $\cos\theta$ and even in $\sin\theta$. Since $a_k\geqslant 0$ for every k, then $f(t-\pi/2)$ is odd, and $f(t)\geqslant 0$ for all $t\in(\pi/2,3\pi/2)$.

The function g is the product of $\sin\theta$ by a function which is even with respect to 0. Hence g is odd with respect to 0.

Finally, every h_{ij} is odd in $\cos\theta$ and even in $\sin\theta$, and thus h_{ij} is odd. Moreover, since $c_{ij}\geqslant 0$, then $h_{ij}(t)\geqslant 0$ for all $t\in(0,\pi)$. Applying Corollary 2.7, (3.3) has no positive periodic solutions, and thus (1.4) has no limit cycles. □

Proposition 3.3. *Consider*

$$F(x,y)=\sum_{k=0}^{(n+1)/2}\left(a_k x^{2k+1}y^{n-2k-1}+b_k x^{n-2k-1}y^{2k+1}\right)+\sum_{2i+2j+2>n}c_{ij}x^{2i+1}y^{2j+1},$$

and assume that n is odd, and $a_k, b_k, c_{ij}\geqslant 0$ for every i, j, k, the inequalities being strict for some i, j, k. Then (1.4) *has no limit cycles.*

The same result holds if the second sum is over $n>2i+2j+2$, $a_k\geqslant 0$ for every k, $b_k\geqslant 0$ for every k, or $c_k\geqslant 0$ for every k, the inequalities being strict for some indices.

Proof. Arguing as in Proposition 3.2, one has that (3.3) satisfies Corollary 2.12 and then (1.4) has no limit cycles. □

When F consists of only three terms, Propositions 3.1, 3.2, and 3.3 may be summarized in the following result:

Theorem 3.4. *For given $n, m, k_1, k_2, k_3\in\mathbb{N}$ and $a, b, c\in\mathbb{R}$, consider the family*

$$\begin{cases} x'=y+x\left(ax^{k_1}y^{n-k_1}+bx^{k_2}y^{n-k_2}+cx^{k_3}y^{m-k_3}\right),\\ y'=-x+y\left(ax^{k_1}y^{n-k_1}+bx^{k_2}y^{n-k_2}+cx^{k_3}y^{m-k_3}\right).\end{cases}\tag{3.5}$$

Then

(i) *If n is odd and m is even then* (3.5) *either has no limit cycles or is a center.*
(ii) *If n and m are odd and k_1 or k_2 have the same parity as k_3 then* (3.5) *either has no limit cycles or is a center.*

Proof. If $c = 0$, proceeding as in Theorem 2.1, by direct integration of (3.3) one obtains that (3.5) is a center. Thus one may assume that $c \neq 0$.

First, assume that m is even. If k_3 is even, then (3.5) satisfies Proposition 3.1. Thus (3.5) has no limit cycles.

If k_3 is odd, there are the following possibilities:

(a) Both k_1 and k_2 are odd. By Proposition 2.9, all solutions of (1.5) are periodic, so that (3.5) is a center. The same holds if $a = 0$ and k_2 odd, $b = 0$ and k_1 odd, or $a = b = 0$.
(b) Either k_1 is even and k_2 is odd, or k_1 is odd and k_2 is even. Since (3.5) satisfies the hypotheses of Proposition 3.3, it has no periodic solutions. If a or b are zero is included in one of the other two cases.
(c) Both k_1 and k_2 are even. Exchanging x and y, one obtains k_1 and k_2 odd. Then, proceeding as in the first case, (3.5) is a center. The same holds if $a = 0$ and k_1 odd, $b = 0$ and k_2 odd, or $a = b = 0$.

Finally, assume that m and k_3 are odd (if k_3 is even, exchanging x and y one obtains the case k_3 odd again). Since n is odd, one of k_1 and $n - k_1$ is even and the other is odd. The same holds for k_2 and $n - k_2$. By hypothesis, k_1 and k_2 cannot both be even, so that there are the following possibilities:

(a) Both k_1 and k_2 are odd. By Proposition 2.9, all solutions of (1.5) are periodic, so that (3.5) is a center. The same holds if $a = 0$ and k_2 odd, or k_1 odd and $b = 0$.
(b) Either k_1 is even, $a \neq 0$, and k_2 is odd, or k_1 is odd, $b \neq 0$, and k_2 is even. Then (3.5) satisfies the hypotheses of Proposition 3.2, and thus has no periodic solutions. $\square$

In [9] the problem of the number of limit cycles of (1.4) was also considered for the case of $F(x, y)$ being a polynomial of degree 2. In that work there appear some open problems when the homogeneous part of degree 2 can change sign. The normal form in that case is

$$\begin{cases} x' = y + x\big(a + bx + cy + dx^2 + exy\big), \\ y' = -x + y\big(a + bx + cy + dx^2 + exy\big) \end{cases}$$

with $d = 0$ or $d = 1$. A corollary of the foregoing theorem is that there are no limit cycles in the case $a = d = 0$.

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